

Problem Set: The Solow–Swan Growth Model

Advanced Macroeconomics — Dr Lei Pan — Total: 100 Marks

Instructions. Answer all questions. Show all mathematical derivations clearly. Answers without derivation receive limited credit. Unless otherwise stated, assume

$$0 < \alpha < 1, \quad 0 < s < 1, \quad n \geq 0, \quad g \geq 0, \quad 0 < \delta < 1, \quad D \equiv (1+n)(1+g), \quad b \equiv D - (1-\delta) = n + g + ng + \delta.$$

The standard Solow approximation ignores the small second-order term ng , so that $b \approx n + g + \delta$.

Question 1: Equilibrium Dynamics, Steady State, and Balanced Growth [Total: 60 marks]

Consider the discrete-time Solow–Swan economy

$$Y_t = F(K_t, A_t L_t) = K_t^\alpha (A_t L_t)^{1-\alpha}, \quad A_{t+1} = (1+g)A_t, \quad L_{t+1} = (1+n)L_t, \\ K_{t+1} = (1-\delta)K_t + I_t, \quad C_t = (1-s)Y_t, \quad S_t = sY_t, \quad I_t = S_t.$$

Define

$$k_t \equiv \frac{K_t}{A_t L_t}, \quad y_t \equiv \frac{Y_t}{A_t L_t}, \quad c_t \equiv \frac{C_t}{A_t L_t}.$$

- (a) Derive the intensive-form production function and the law of motion

$$k_{t+1} = \mathcal{G}(k_t).$$

Then derive $k_{t+1} - k_t$ and explain the economic meaning of bk_t . [15 marks]

- (b) Solve explicitly for the positive steady state k^* , y^* , and c^* . Then compute the exact and approximate steady-state values when

$$\alpha = \frac{1}{3}, \quad s = 0.24, \quad n = 0.01, \quad g = 0.02, \quad \delta = 0.05.$$

[15 marks]

- (c) Prove that the steady state is unique and globally stable for any $k_0 > 0$. Then derive the local convergence coefficient

$$\lambda \equiv \mathcal{G}'(k^*).$$

Compute λ and the half-life of deviations from steady state for the numerical parameters in part (b). [18 marks]

- (d) Derive the balanced-growth-path growth rates of K_t , Y_t , C_t , I_t , K_t/L_t , and Y_t/L_t . Then derive the comparative statics of k^* and y^* with respect to s , n , g , and δ . [12 marks]

Question 2: Golden Rule, Dynamic Inefficiency, and Saving-Rate Policy [Total: 40 marks]

Continue with the Cobb–Douglas Solow economy from Question 1:

$$y = f(k) = k^\alpha, \quad c^* = (1-s)f(k^*), \quad sf(k^*) = bk^*.$$

- (a) Define the Golden Rule level of capital as the level of k that maximises sustainable steady-state consumption:

$$c(k) = f(k) - bk.$$

Derive k_{GR} , y_{GR} , c_{GR} , and the saving rate s_{GR} that decentralises the Golden Rule steady state. [12 marks]

- (b) Using

$$\alpha = \frac{1}{3}, \quad s_0 = 0.24, \quad n = 0.01, \quad g = 0.02, \quad \delta = 0.05,$$

compute k_{GR} , y_{GR} , c_{GR} , and s_{GR} . Compare the initial economy with $s_0 = 0.24$ to the Golden Rule allocation. Is the economy saving too much or too little? [10 marks]

- (c) Suppose the government permanently raises the saving rate from $s_0 = 0.24$ to $s_1 = 0.40$. Compute the immediate effect on consumption per unit of effective labour, holding k fixed at the old steady state. Then compute the new steady-state values k_1^* , y_1^* , and c_1^* . Does the policy raise long-run consumption? Does it achieve the Golden Rule? [10 marks]

(d) Derive the local transition dynamics around the new steady state after the saving-rate change. Let

$$\widehat{k}_t \equiv \ln k_t - \ln k_1^*.$$

Show that locally $\widehat{k}_{t+1} \approx \lambda \widehat{k}_t$, and derive the approximate growth rate of output per worker during transition.
[8 marks]

Detailed Solutions

Solution to Question 1

[60 marks]

Part (a)

[15 marks]

The aggregate production function is

$$Y_t = K_t^\alpha (A_t L_t)^{1-\alpha}.$$

Divide both sides by $A_t L_t$:

$$\frac{Y_t}{A_t L_t} = \frac{K_t^\alpha (A_t L_t)^{1-\alpha}}{A_t L_t} = K_t^\alpha (A_t L_t)^{-\alpha} = \left(\frac{K_t}{A_t L_t} \right)^\alpha.$$

Therefore,

$$\boxed{y_t = f(k_t) = k_t^\alpha.}$$

Goods-market clearing gives

$$I_t = S_t = sY_t.$$

Substitute into the capital-accumulation equation:

$$K_{t+1} = (1 - \delta)K_t + sY_t.$$

Using the production function:

$$K_{t+1} = (1 - \delta)K_t + sK_t^\alpha (A_t L_t)^{1-\alpha}.$$

Divide by $A_t L_t$:

$$\frac{K_{t+1}}{A_t L_t} = (1 - \delta)k_t + sk_t^\alpha.$$

The left-hand side is

$$\frac{K_{t+1}}{A_t L_t} = \frac{K_{t+1}}{A_{t+1} L_{t+1}} \frac{A_{t+1} L_{t+1}}{A_t L_t} = Dk_{t+1}.$$

Hence,

$$Dk_{t+1} = (1 - \delta)k_t + sk_t^\alpha.$$

Therefore,

$$\boxed{k_{t+1} = \mathcal{G}(k_t) = \frac{(1 - \delta)k_t + sk_t^\alpha}{D}.$$

Subtract k_t from both sides:

$$k_{t+1} - k_t = \frac{(1 - \delta)k_t + sk_t^\alpha - Dk_t}{D} = \boxed{\frac{sk_t^\alpha - bk_t}{D}}.$$

The term sk_t^α is actual investment per unit of effective labour. The term bk_t is break-even investment per unit of effective labour: the investment required to keep k_t constant despite population growth, technological progress, the interaction term ng , and depreciation.

Marking guide: intensive-form production function, 3; transition equation, 5; expression for $k_{t+1} - k_t$, 4; interpretation of bk_t , 3.

Part (b)

[15 marks]

A steady state satisfies

$$k_{t+1} = k_t = k^*.$$

Using the transition equation:

$$k^* = \frac{(1 - \delta)k^* + s(k^*)^\alpha}{D}.$$

Multiply by D :

$$Dk^* = (1 - \delta)k^* + s(k^*)^\alpha.$$

Rearrange:

$$[D - (1 - \delta)]k^* = s(k^*)^\alpha.$$

Since $b = D - (1 - \delta)$,

$$bk^* = s(k^*)^\alpha.$$

For $k^* > 0$:

$$b(k^*)^{1-\alpha} = s.$$

Therefore,

$$k^* = \left(\frac{s}{b}\right)^{\frac{1}{1-\alpha}}.$$

Then

$$y^* = (k^*)^\alpha = \left(\frac{s}{b}\right)^{\frac{\alpha}{1-\alpha}}, \quad c^* = (1-s)y^* = (1-s)\left(\frac{s}{b}\right)^{\frac{\alpha}{1-\alpha}}.$$

Using

$$\alpha = \frac{1}{3}, \quad s = 0.24, \quad n = 0.01, \quad g = 0.02, \quad \delta = 0.05,$$

we have

$$b = n + g + ng + \delta = 0.0802.$$

Since

$$\frac{1}{1-\alpha} = \frac{3}{2}, \quad \frac{\alpha}{1-\alpha} = \frac{1}{2},$$

$$k^* = \left(\frac{0.24}{0.0802}\right)^{3/2} \approx 5.1767,$$

$$y^* = (k^*)^{1/3} \approx 1.7299, \quad c^* = 0.76(1.7299) \approx 1.3147.$$

Using the approximation $\tilde{b} = n + g + \delta = 0.08$:

$$\tilde{k}^* = \left(\frac{0.24}{0.08}\right)^{3/2} = 3^{3/2} \approx 5.1962,$$

$$\tilde{y}^* \approx 1.7321, \quad \tilde{c}^* \approx 1.3164.$$

Marking guide: steady-state equation, 3; closed-form k^* , 4; y^* and c^* , 3; exact values, 3; approximate values, 2.

Part (c)

[18 marks]

The steady state solves

$$bk = sk^\alpha,$$

or

$$sk^{\alpha-1} = b.$$

Because $0 < \alpha < 1$, $\alpha - 1 < 0$, so $k^{\alpha-1}$ is strictly decreasing in k . Moreover,

$$\lim_{k \rightarrow 0^+} sk^{\alpha-1} = +\infty, \quad \lim_{k \rightarrow \infty} sk^{\alpha-1} = 0.$$

Since $b > 0$, there exists a unique positive steady state.

For stability:

$$k_{t+1} - k_t = \frac{k_t(sk_t^{\alpha-1} - b)}{D}.$$

At k^* ,

$$s(k^*)^{\alpha-1} = b.$$

If $k_t < k^*$, then $k_t^{\alpha-1} > (k^*)^{\alpha-1}$, so

$$sk_t^{\alpha-1} > b,$$

and hence $k_{t+1} - k_t > 0$. If $k_t > k^*$, then

$$sk_t^{\alpha-1} < b,$$

so $k_{t+1} - k_t < 0$. Therefore, k_t converges to k^* from any $k_0 > 0$.

The transition function is

$$\mathcal{G}(k) = \frac{(1-\delta)k + sk^\alpha}{D}.$$

Differentiate:

$$\mathcal{G}'(k) = \frac{(1-\delta) + s\alpha k^{\alpha-1}}{D}.$$

Evaluate at k^* :

$$\lambda = \mathcal{G}'(k^*) = \frac{(1-\delta) + s\alpha(k^*)^{\alpha-1}}{D}.$$

Using $s(k^*)^{\alpha-1} = b$:

$$\lambda = \frac{(1 - \delta) + \alpha b}{D}.$$

Numerically,

$$D = (1.01)(1.02) = 1.0302, \quad b = 0.0802.$$

Thus,

$$\lambda = \frac{0.95 + \frac{1}{3}(0.0802)}{1.0302} \approx 0.9481.$$

The half-life h solves $\lambda^h = 1/2$:

$$h = \frac{\ln(1/2)}{\ln \lambda} = \frac{\ln(0.5)}{\ln(0.9481)} \approx 13.01.$$

Marking guide: uniqueness proof, 4; global stability, 6; derivative, 3; expression for λ , 2; numerical λ and half-life, 3.

Part (d)

[12 marks]

On the balanced growth path, $k_t = k^*$. Since

$$K_t = A_t L_t k^*,$$

$$\frac{K_{t+1}}{K_t} = \frac{A_{t+1} L_{t+1}}{A_t L_t} = (1 + g)(1 + n) = D.$$

Thus, K_t grows at exact net rate $D - 1 = n + g + ng$. Since

$$Y_t = A_t L_t y^*,$$

Y_t also grows at rate $n + g + ng$. Since $C_t = (1 - s)Y_t$ and $I_t = sY_t$, both C_t and I_t grow at the same rate.

Capital per worker is

$$\frac{K_t}{L_t} = A_t k^*,$$

so it grows at rate g . Output per worker is

$$\frac{Y_t}{L_t} = A_t y^*,$$

so it also grows at rate g .

For comparative statics:

$$k^* = \left(\frac{s}{b}\right)^{\frac{1}{1-\alpha}},$$

so

$$\ln k^* = \frac{1}{1 - \alpha} (\ln s - \ln b).$$

Therefore,

$$\frac{\partial \ln k^*}{\partial \ln s} = \frac{1}{1 - \alpha} > 0.$$

Since

$$b = n + g + ng + \delta,$$

$$\frac{\partial b}{\partial n} = 1 + g, \quad \frac{\partial b}{\partial g} = 1 + n, \quad \frac{\partial b}{\partial \delta} = 1.$$

Hence,

$$\frac{\partial \ln k^*}{\partial n} = -\frac{1 + g}{(1 - \alpha)b} < 0,$$

$$\frac{\partial \ln k^*}{\partial g} = -\frac{1 + n}{(1 - \alpha)b} < 0,$$

and

$$\frac{\partial \ln k^*}{\partial \delta} = -\frac{1}{(1 - \alpha)b} < 0.$$

Because $y^* = (k^*)^\alpha$,

$$\frac{\partial \ln y^*}{\partial x} = \alpha \frac{\partial \ln k^*}{\partial x}, \quad x \in \{s, n, g, \delta\}.$$

Marking guide: BGP growth rates, 5; comparative statics for k^* , 5; comparative statics for y^* , 2.

Solution to Question 2**[40 marks]****Part (a)****[12 marks]**

The Golden Rule level of capital maximises

$$c(k) = f(k) - bk.$$

Since $f(k) = k^\alpha$:

$$c(k) = k^\alpha - bk.$$

The first-order condition is

$$\frac{dc(k)}{dk} = 0.$$

Since

$$\frac{dc(k)}{dk} = \alpha k^{\alpha-1} - b,$$

the Golden Rule condition is

$$\alpha k_{\text{GR}}^{\alpha-1} = b.$$

Equivalently,

$$k_{\text{GR}}^{1-\alpha} = \frac{\alpha}{b}.$$

Thus,

$$k_{\text{GR}} = \left(\frac{\alpha}{b}\right)^{\frac{1}{1-\alpha}}.$$

Golden Rule output is

$$y_{\text{GR}} = \left(\frac{\alpha}{b}\right)^{\frac{\alpha}{1-\alpha}}.$$

Golden Rule consumption is

$$c_{\text{GR}} = f(k_{\text{GR}}) - bk_{\text{GR}}.$$

Using $bk_{\text{GR}} = \alpha k_{\text{GR}}^\alpha = \alpha y_{\text{GR}}$:

$$c_{\text{GR}} = (1 - \alpha)y_{\text{GR}}.$$

The saving rate that decentralises the Golden Rule satisfies

$$s_{\text{GR}}f(k_{\text{GR}}) = bk_{\text{GR}}.$$

Since $bk_{\text{GR}} = \alpha f(k_{\text{GR}})$,

$$s_{\text{GR}} = \alpha.$$

Marking guide: optimisation problem, 2; FOC, 3; k_{GR} and y_{GR} , 3; c_{GR} , 2; $s_{\text{GR}} = \alpha$, 2.

Part (b)**[10 marks]**

Given

$$\alpha = \frac{1}{3}, \quad n = 0.01, \quad g = 0.02, \quad \delta = 0.05,$$

$$b = n + g + ng + \delta = 0.0802.$$

Thus,

$$k_{\text{GR}} = \left(\frac{1/3}{0.0802}\right)^{3/2} \approx 8.4734.$$

Then

$$y_{\text{GR}} = k_{\text{GR}}^{1/3} \approx 2.0387.$$

Golden Rule consumption is

$$c_{\text{GR}} = \frac{2}{3}(2.0387) \approx 1.3591.$$

The Golden Rule saving rate is

$$s_{\text{GR}} = \frac{1}{3} \approx 0.3333.$$

With $s_0 = 0.24$:

$$k_0^* \approx 5.1767, \quad y_0^* \approx 1.7299, \quad c_0^* \approx 1.3147.$$

Since

$$s_0 = 0.24 < s_{GR} = \frac{1}{3},$$

the economy saves too little relative to the Golden Rule. Equivalently, $k_0^* < k_{GR}$.

Marking guide: b , 1; k_{GR} , 2; y_{GR} and c_{GR} , 3; s_{GR} , 1; comparison and interpretation, 3.

Part (c)

[10 marks]

When the saving rate rises from $s_0 = 0.24$ to $s_1 = 0.40$, capital is predetermined in the short run. Therefore, output per effective worker is initially unchanged:

$$y_0^* \approx 1.7299.$$

Initial consumption is

$$c_0^* = (1 - s_0)y_0^* = 0.76(1.7299) \approx 1.3147.$$

Immediate consumption after the policy change is

$$c_{\text{impact}} = (1 - s_1)y_0^* = 0.60(1.7299) \approx 1.0379.$$

Thus, consumption falls immediately.

The new steady state is

$$k_1^* = \left(\frac{s_1}{b}\right)^{\frac{1}{1-\alpha}} = \left(\frac{0.40}{0.0802}\right)^{3/2} \approx 11.1385.$$

Then

$$y_1^* = (k_1^*)^{1/3} \approx 2.2333.$$

New steady-state consumption is

$$c_1^* = (1 - s_1)y_1^* = 0.60(2.2333) \approx 1.3400.$$

Since $1.3400 > 1.3147$, the policy raises long-run consumption relative to the initial steady state. However,

$$s_1 = 0.40 > s_{GR} = \frac{1}{3},$$

so the policy overshoots the Golden Rule.

Marking guide: impact logic, 2; impact consumption, 2; new steady-state values, 3; long-run comparison, 1; Golden Rule interpretation, 2.

Part (d)

[8 marks]

After the saving-rate change:

$$k_{t+1} = \mathcal{G}_1(k_t) = \frac{(1 - \delta)k_t + s_1 k_t^\alpha}{D}.$$

The new steady state satisfies $k_1^* = \mathcal{G}_1(k_1^*)$. Define

$$\widehat{k}_t = \ln k_t - \ln k_1^*.$$

A first-order log-linearisation around k_1^* gives

$$\widehat{k}_{t+1} \approx \lambda \widehat{k}_t,$$

where

$$\lambda = \mathcal{G}'_1(k_1^*).$$

Since

$$\mathcal{G}'_1(k) = \frac{(1 - \delta) + s_1 \alpha k^{\alpha-1}}{D},$$

$$\lambda = \frac{(1 - \delta) + s_1 \alpha (k_1^*)^{\alpha-1}}{D}.$$

Using $s_1 (k_1^*)^{\alpha-1} = b$:

$$\lambda = \frac{(1 - \delta) + \alpha b}{D}.$$

Numerically,

$$\lambda = \frac{0.95 + \frac{1}{3}(0.0802)}{1.0302} \approx 0.9481.$$

Output per worker is

$$\frac{Y_t}{L_t} = A_t k_t^\alpha.$$

Taking logs:

$$\ln\left(\frac{Y_t}{L_t}\right) = \ln A_t + \alpha \ln k_t.$$

Therefore,

$$\ln\left(\frac{Y_{t+1}/L_{t+1}}{Y_t/L_t}\right) = \ln(1+g) + \alpha(\ln k_{t+1} - \ln k_t).$$

For small g :

$$g_{Y/L,t} \approx g + \alpha(\ln k_{t+1} - \ln k_t).$$

During the transition, $k_t < k_1^*$, so k_t rises and output per worker grows faster than g . As $k_t \rightarrow k_1^*$, output-per-worker growth returns to g .

Marking guide: transition equation, 1; log-linearisation, 2; λ , 2; numerical λ , 1; transitional output-per-worker growth, 2.